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ON THE COMMONALITY OF EQUATIONS FOR NATURAL CONVECTION FROM IMMERSED BODIES

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NOMENCLATURE

A,	surface area of heater;
c_p,	heat capacity of fluid;
D,	diameter of a sphere or cylinder;
F_B, F_D,	total buoyant force on heater and total drag force on heater;
g,	acceleration of gravity;
h,	average heat transfer coefficient;
k,	thermal conductivity of fluid;
L,	distance traveled by the boundary layer from bottom to top of heater;
Nu	Nusselt number, hL/k;
Pr,	Prandtl number, $\mu c_p/k$;
Ra,	Rayleigh number, $g\beta\Delta T\rho^2 c_p L^3/\mu k$;
$T, \bar{T}, T_\infty$,	local temperature, overbar denotes average of T over y in b.l., subscript denotes T in fluid bulk;
$u, \bar{u}$,	local velocity in a boundary layer, overbar denotes average of u over y in b.l.;
u_x,	a characteristic velocity used by Eckert (not used in the present study);
y,	distance normal to heater wall;
β,	coefficient of thermal expansion;
δ,	boundary layer thickness;
δ_S,	thickness of effective shear layer;
δ_T,	thickness of layer with energy equal to that in the boundary layer;
ΔT,	difference between temperature of wall and temperature of fluid bulk.
μ,	viscosity;
ρ,	density of fluid.

A CURIOUS fact about laminar natural convection is that, regardless of the shape of the heater, the overall heat transfer expression

$$Nu/Ra^{\frac{1}{4}} \simeq \tfrac{1}{2}$$

applies to wholly immersed isothermal bodies, within a few per cent.

We shall seek to explain the constancy of $Nu/Ra^{\frac{1}{4}}$ by estimating the heat exchange with a general isothermal body. The estimate will be approximate in that certain details of the boundary layer will be lumped. It will proceed from the fact that the overall buoyant force exerted on the body by the boundary layer must be balanced by the overall viscous drag force exerted by the body on the boundary layer. In doing this it is reasonable to ignore inertia since inertia is known to play a minor role in shaping the boundary layer when $Pr = \mathcal{O}(1)$, and this role vanishes as Pr increases.

The convective boundary layer (b.l.) will be lumped into an isothermal slab, uniformly at the wall temperature, moving at a uniform average velocity, $\bar{u}$. It will have a thickness, $\delta_T < \delta$, such as to give it the same buoyancy as the actual b.l. And it will be located at a distance, δ_S, away from the wall such as to exert the same shear stress on the wall as the actual b.l. does. Then, for a body of surface area, A, the total forces can be approximated as:

$$\text{drag of body on the b.l.: } F_D = \mu A \left.\frac{\partial u}{\partial y}\right|_{\text{wall}} = \mu A \frac{\bar{u}}{\delta_S} \quad (1)$$

$$\text{buoyancy of b.l. on the body, } F_B = \rho g \beta \Delta T (A\delta_T). \quad (2)$$

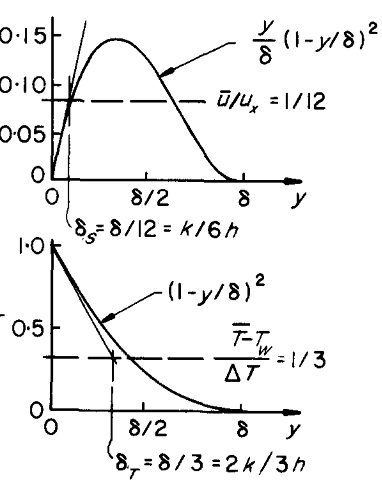

FIG. 1. Eckert's velocity and temperature profiles for a flat plate.

We must now form estimates of $\bar{u}$, δ_T and δ_S. A simple energy balance gives

$$\bar{u} = hL/\rho c_p \delta_T. \tag{3}$$

The estimation of δ_T and δ_S requires some knowledge of actual velocity and temperature profiles. We therefore look for reasonable approximate profiles that do not depend on the particular shape of the heater. The profiles first used by Eckert [1] for the vertical plate fit this requirement as long as $Pr \geqslant \mathcal{O}(1)$, and have subsequently been used for a variety of configurations (see e.g. [2]). These profiles are shown in Fig. 1. Since the mean of $(T - T_\infty)/\Delta T$ is $\frac{1}{3}$, δ_T is equal to $\delta/3$. Eckert's velocity profile is nondimensionalized with a characteristic velocity, u_x, which, although it depends upon heater geometry in his formulation, will not enter our computations. The wall velocity gradient intersects the mean velocity ($\bar{u}/u_x = \frac{1}{12}$) at $y = \delta_S = \delta/12$.

Finally, since for this temperature profile $\delta = 2k/h$, we obtain

$$\delta_T = \frac{2}{3}\frac{k}{h};\ \delta_S = \frac{1}{6}\frac{k}{h}, \quad \text{and} \quad \bar{u} = \frac{3}{2}\frac{Lk}{2(k/h)^2 \rho c_p}. \tag{4}$$

It follows from equations (1), (2) and (4) that

$$F_D = 9\frac{\mu k}{c_p}\left(\frac{h}{k}\right)^3 AL; \quad F_B = \frac{2}{3}\rho g \beta \Delta T A\left(\frac{k}{h}\right). \tag{5}$$

Equating these forces and simplifying the result yields

$$Nu/Ra^{\frac{1}{4}} = 0{\cdot}52 \tag{6}$$

for natural convection, where Nu and Ra are based on the distance, L, that a particle travels in the b.l.

Table 1 lists some typical theoretical and experimental values of $Nu/Ra^{\frac{1}{4}}$ for four configurations. The experimental values are all for the laminar regime at sufficiently large Rayleigh numbers to assure that a boundary layer exists. Except for the flat plate results, these values appear to be high since they have been written in terms of longer characteristic lengths than are customary. It is interesting to note that, had an inertial force been included in the force balance, there would have resulted an unimportant correction term (with Pr in it) similar to that given by Eckert as shown in Table 1. The relatively minor contribution of inertia decreases with increasing Pr as this factor suggests.

The data in each case differ very little from equation (6). Thus a consideration of the common features of the natural convection problem leads to an explanation as to why $Nu/Ra^{\frac{1}{4}}$ is so nearly constant. It also suggests the use of the *length of travel* of the b.l. as the appropriate characteristic length in the group. We recommend the use of equation (6) with this characteristic length to predict the heat transfer from any submerged isothermal body, for $Pr \geqslant \mathcal{O}(1)$.

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Table 1. Predicted and observed values of Nu/Ra$^{\frac{1}{4}}$ *for various isothermal submerged bodies (laminar regime)*

Configuration	$Nu/Ra^{\frac{1}{4}}$ Theoretical		Experimental	
vertical plate (L = height)	$0{\cdot}678\,[Pr/(0{\cdot}952 + Pr)]^{\frac{1}{4}}$ = 0·55 for air	[1]	0·55	[3]
	0·503 for high Pr	[11]		
horizontal circular cylinder $L = \frac{\pi}{2}D$	0·45 to 0·48	[2, 4, 5]	0·59	[3]
			0·51	[6]
sphere $L = \frac{\pi}{2}D$	0·53	[7]	0·575	[8]
			0·48	[9]
horizontal square cylinder on one edge [L = 2 (side)]	—		0·48	[10]

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